

Probability and Statistics

unit 1:

Random Variables & Probability Distributions

- Random variables (R.V)
- Discrete random variables and continuous random variables
- Probability mass and probability density function.
- Expectation mean and variance
- Properties and problems.

⇒ Probability Distributions.

- Binomical
- Poisson
- Gaussian (or) normal

Properties and problems

Sample Space: The set of all possible outcomes in a Random experiment is called sample space and is denoted by S .

ex 1: In tossing of coin, The sample space.

$$S = \{H, T\}$$

eg 2: In throwing of die, The sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

eg 3: In throwing of coins simultaneously, the sample space

$$S = \{HH, HT, TH, TT\}$$

Note 1: In tossing of n coins simultaneously, The sample space S contains [the sample space] $\times 2^n$ elements.

$$n(S) = 2^n$$

Note 2: In throwing of n dice, The sample space S contains 6^n elements.

$$n(S) = 6^n$$

Event:

Any subset of a sample space S , This is called an event and $\emptyset$ is denoted by E .

Eg: In tossing of a coin, $S = \{H, T\}$

The following are events of S .

$$S = \{H, T\}$$

$$\emptyset = \{H\}, \{T\}, \{H, T\}$$

Probability:

Probability of an event (E) is denoted by $P(E)$ and is defined by,

$$P(E) = \frac{\text{Favourable outcomes of } E}{\text{Total no. of outcomes}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

Note 1: Probability of non-occurrence of an event E denoted by $P(\bar{E})$ (or) $P(E^c)$ and is defined by

$$P(\bar{E}) = P(E^c) = 1 - P(E)$$

Note 2:

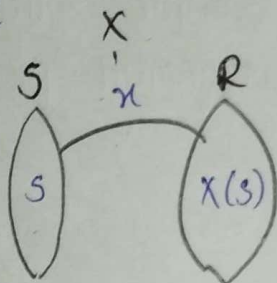
$$P(\bar{E}) = P(E^c) + P(E) = 1$$

Random Variables:

A Random Variable (X) is a function from the sample space (S) to the set of real numbers (R) i.e.,

$$X: S \rightarrow R$$

which assign each point s of S to a real number $X(s)$ of ' R '



Notation:

If Random Variable is denoted by X , Then ' x ' denotes one of its values.

Note:

Random Variable (R.V) are 2 types. They are;

i) Discrete R.V

ii) Continuous R.V.

i) Discrete R.V.

A random variable X is said to be discrete R.V, if the corresponding sample space ' S ' is finite or countable.

eg: Tossing a coin, Throwing a die, The no. of defectives in a sample of electric bulbs, The no. of printing mistakes in each page of a book, etc.

ii) Continuous R.V.

A random variable X is said to be continuous R.V, if the sample space ' S ' contains infinite numbers.

eg: The height, age & weight of individuals

Note 1: A Continuous R.V that takes any value within the limits of the variable.

Note 2: A Discrete R.V represents the no. of distinct values that can be counted/compared on any event.

Probability Distributions

A probability distribution is a function or rule that describes how the probabilities are assigned to different possible outcomes of a random experiment.

Probability Distribution are of 2 types,

- i) Discrete probability distribution
- ii) Continuous probability distribution.

i) Discrete probability distribution:

Each event in a sample space 'S' has certain probability of occurrence. A formula representing all these probabilities which a discrete probability distribution

eg: In tossing of two coins simultaneously.

the sample space $S = \{HH, TH, HT, TT\}$

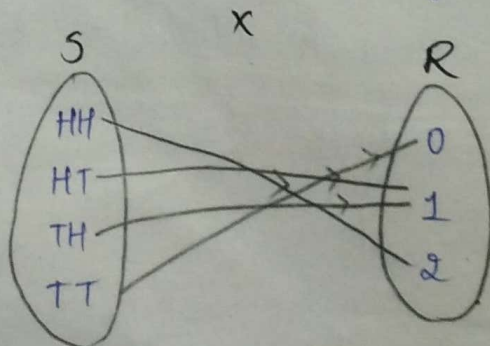
For every event x in 'S'. The R.V X defined to be no. of heads.

i.e., $P(X=x) = \text{No. of heads}$.

Now $P(X=0) = \text{probability of getting no. of heads} = P\{TT\} = \frac{1}{4}$

$P(X=1) = \text{probability of getting 1 head} = P\{HT, TH\} = \frac{2}{4}$

$P(X=2) = \text{probability of getting 2 heads} = P\{HH\} = \frac{1}{4}$



The discrete probability is given by,

$X=x$	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

(or)

$$(0, \frac{1}{4}), (1, \frac{2}{4}), (2, \frac{1}{4})$$

Note :

The sum of all the probabilities in a discrete probability Distribution.

$$\text{i.e., } \sum P(X=x) = 1$$

Probability mass function (or) Mass function.

For a discrete random variable X , the function f

$$f(x) = P(X=x)$$

satisfies the following is known as Probability mass function (or) mass function.

$$\text{i) } f(x) = P(X=x) \geq 0 \quad \forall x$$

$$\text{ii) } \sum_i f(x_i) = \sum_i P(X=x_i) = 1$$

Continuous Probability Distribution.

If the Random Variable X is continuous, then the corresponding probability Distribution is known as Continuous probability Distribution

Probability Density Function or Density Function.

For a continuous Random Variable X , the function

$$f(x) = P(X=x) \geq 0 \quad \forall x$$

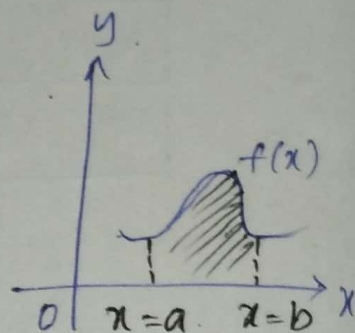
Satisfies the following is known as Probability Density Function or simply Density function.

$$i) f(x) = P(X=x) \geq 0 \quad \forall x$$

$$ii) \int_{-\infty}^{\infty} f(x) dx = 1$$

$$iii) P(a < X < b) = \int_a^b f(x) dx$$

(area under $f(x)$ between the ordinates $x=a, x=b$)



Note 1)

For a continuous R.V X ,

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$$

Note 2)

For a Discrete R.V X , the above 4 probabilities are not equivalent.

Mean or Expectation or Expected Values of a R.V " X "

Mean or Expectation or Expected values are denoted by M (or) $E(X)$

$$\mu = E(X) = \begin{cases} \sum_i x_i f(x_i) = \sum_i x_i P(X=x_i), & \text{if } X \text{ is Discrete R.V} \\ \int_{-\infty}^{\infty} x \cdot f(x) dx, & \text{if } X \text{ is Continuous R.V} \end{cases}$$

Note :

For a Discrete R.V " X ", the sum of product of R.V and their corresponding probabilities is its Mean.

Variance:

Variance of a R.V "X" is denoted by

$$\text{Var}(X) \text{ (or) } \sigma^2 \text{ (or) } E(X - \mu)^2$$

and is defined by

$$\text{Var}(X) = \sigma^2 = E(X - \mu)^2 = \begin{cases} \sum_i (x_i - \mu)^2 \cdot f(x_i), & \text{if } X \text{ is Discrete R.V.} \\ \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, & \text{if } X \text{ is Continuous R.V.} \end{cases}$$

Standard Deviation

Positive square root of Variance is known as Standard Deviation and it is denoted by σ

i.e., $\boxed{S.D = \sigma}$

Note:

$$1) E(X^2) = \begin{cases} \sum x_i^2 f(x_i) = \sum x_i^2 \cdot P(X = x_i) & \text{if } X \text{ is Discrete R.V.} \\ \int_{-\infty}^{\infty} x^2 f(x) dx, & \text{if } X \text{ is Continuous R.V.} \end{cases}$$

2) If X is a Discrete R.V., then Variance of X

$$\text{Var}(X) = \sigma^2 = E(X - \mu)^2 = E(X^2) - [E(X)]^2$$

$$\text{Sol: } \sigma^2 = E(X - \mu)^2 = \sum (x - \mu)^2 \cdot f(x)$$

$$= \sum (x^2 - 2\mu \cdot x + \mu^2) \cdot f(x)$$

$$= \sum x^2 f(x) - \sum 2\mu \cdot x f(x) + \sum \mu^2 \cdot f(x)$$

$$= \sum x^2 f(x) - 2\mu \sum x f(x) + \mu^2 \sum f(x)$$

$$= E(X^2) - 2\mu \cdot E(X) + \mu^2 \cdot 1$$

$$(\because \sum x \cdot f(x) = E(X))$$

$$\sum f(x) = \sum P(X = x) = 1$$

$$= E(X^2) - 2\mu \cdot \mu + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

$$\sigma^2 = E(X - \mu^2) = E(X^2) - [E(X)]^2$$

Note:

If X is a Continuous R.V, then above formula is also true for variance.

i.e., if X is a continuous R.V.

then variance $\boxed{\sigma^2 = E(X^2) - [E(X)]^2}$

Q1) If X is a R.V and k is a constant then prove that

$$i) E(X+k) = E(X) + k$$

$$ii) E(kX) = kE(X)$$

$$iii) E(X+Y) = E(X) + E(Y)$$

Note:

If X is a R.V means by default it is Discrete Random Variable.

Sol

We know that for discrete R.V X

$$\text{mean } \mu = E(X) = \sum x \cdot f(x) = \sum x \cdot P(X=x)$$

$$\therefore E(X+k) = \sum (x+k) \cdot f(x)$$

$$= \sum [x \cdot f(x) + k \cdot f(x)]$$

$$= \sum x \cdot f(x) + \sum k \cdot f(x)$$

$$= \sum x f(x) + k \cdot \sum f(x)$$

$$= E(X) + k \cdot 1$$

$$\boxed{E(X+k) = E(X) + k}$$

[$\because$ for a Discrete R.V X ,

$$\sum f(x) = \sum (P(X=x)) = 1]$$

$$ii) E(KX) = K \cdot E(X)$$

$$E(KX) = \sum (Kx) \cdot f(x) \\ = K \cdot \sum x \cdot f(x)$$

$$\boxed{E(KX) = K \cdot E(X)}$$

$$iii) E(X+Y) = E(X) + E(Y)$$

$$E(X+Y) = \sum (x+y) \cdot f(x)$$

$$= \sum x \cdot f(x) + \sum y \cdot f(x)$$

$$= \sum x \cdot f(x) + \sum y \cdot f(y)$$

$$= E(X) + E(Y)$$

[replace x by y
by probability]

$$\boxed{E(X+Y) = E(X) + E(Y)}$$

2) If X and Y are two independent R.V then

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

2) If X is a R.V and K is a constant then prove that,

$$i) \text{Var}(X+K) = \text{Var}(X)$$

$$ii) \text{Var}(KX) = K^2 \text{Var}(X)$$

Sol.

We know that,

$$\text{Var}(X) = \sigma^2 = E(X^2) - [E(X)]^2$$

$$i) \text{Var}(X+K) = E(X+K)^2 - [E(X+K)]^2$$

$$= \sum (X+K)^2 \cdot f(x) - [E(X+K)]^2$$

$$= \sum (x^2 + 2Kx + K^2) f(x)$$

$$- [E(X) + K]^2$$

$$\left[\begin{aligned} E(X+K) &= \\ E(X) + K \end{aligned} \right]$$

$$= \sum x^2 f(x) + \sum 2Kx f(x) + \sum K^2 f(x)$$

$$- [E(X)]^2 + 2KE(X) + K^2]$$

$$= \sum x^2 \cdot f(x) + 2k \sum x f(x) + k^2 \cdot \sum f(x) - [E(x)]^2 + 2k E(x) + k^2$$

$$= E(x^2) + 2k E(x) + k^2 \cdot 1 - [E(x)]^2 + 2k E(x) + k^2$$

$$= E(x^2) - [E(x)]^2$$

$$\boxed{\text{Var}(X+k) = \text{Var}(X)}$$

$$\text{ii) } \text{Var}(kX) = E(kX)^2 - [E(kX)]^2$$

$$= \sum (kx)^2 \cdot f(x) - [k \cdot E(x)]^2 \quad [\because E(kX) = k \cdot E(x)]$$

$$= k^2 \sum x^2 \cdot f(x) - k^2 [E(x)]^2$$

$$= k^2 [E(x^2) - [E(x)]^2]$$

$$= k^2 \text{Var}(X)$$

$$\boxed{\text{Var}(kX) = k^2 \cdot \text{Var}(X)}$$

Note :

$$1) E(X+2) = E(X) + 2$$

$$2) E(2X+3) = E(2X) + 3 \\ = 2E(X) + 3$$

$$3) E(2X+3Y) = E(2X) + E(3Y) \\ = 2E(X) + 3E(Y)$$

$$4) \text{Var}(X+2) = \text{Var}(X)$$

$$5) \text{Var}(2X+3) = \text{Var}(2X) \\ = 2^2 \text{Var}(X) \\ = 4 \text{Var}(X)$$

3) If X is a R.V and $\text{Var}(X) = 2$, then find $\text{Var}(3X+4)$

Sol. $\text{Var}(3X+4) = \text{Var}(3X)$
 $= 3^2 \text{Var}(X)$ [given $\text{Var}(X) = 2$]
 $= 9 \cdot 2$
 $= 18$

Note:

The probability distribution of a discrete R.V 'X' is given by means of the following:

$X = x_1$	x_1	x_2	x_3	...	x_{i-1}	x_i	x_{i+1}	...	x_n
$f(x_1)$ $= P(X=x_1)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	...	$f(x_{i-1})$	$f(x_i)$	$f(x_{i+1})$	...	$f(x_n)$

Further,

i) $P(X < x_i) = f(x_1) + f(x_2) + \dots + f(x_{i-1})$
 $= P(X = x_1) + P(X = x_2) + \dots + P(X = x_{i-1})$

ii) $P(X \leq x_i) = f(x_1) + f(x_2) + \dots + f(x_i)$
 $= P(X = x_1) + P(X = x_2) + \dots + P(X = x_i)$

iii) $P(X > x_i) = P(X = x_{i+1}) + P(X = x_{i+2}) + \dots + P(X = x_n)$
 $= 1 - [P(X = x_1) + P(X = x_2) + \dots + P(X = x_i)]$

$P(X > x_i) = 1 - P(X \leq x_i)$

[$\because f(x_1) + f(x_2) + \dots + f(x_i) + f(x_{i+1}) + \dots + f(x_n) = 1$
 $f(x_i) + f(x_{i+1}) + \dots + f(x_n) = 1 - [f(x_1) + f(x_2) + \dots + f(x_i)]$
 $1 - [f(x_1) + f(x_2) + \dots + f(x_i)]$]

i) A R.V X has the following probability function (distribution)

$X=x$	0	1	3	4	5	6	7
$f(x) = P(X=x)$	0	K	$2K$	$2K$	$3K$	K^2	$7K^2+K$

- i) Find the value of K ?
ii) Evaluate $P(X < 6)$, $P(X \geq 6)$
iii) Evaluate $P(0 < X < 5)$

Sol.
~~Given~~ dis

Given distribution is Discrete probability distribution

i) We know that, $\sum f(x) = \sum P(X=x) = 1$

$$\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 7K^2 + K = 1$$

$$\Rightarrow 8K^2 + 9K - 1 = 0$$

$$\Rightarrow K = 0.102, K = -1.2$$

$$\Rightarrow \boxed{K = 0.102} \quad [\text{probability can never be zero}]$$

lies b/w 0 to 1.

$$\begin{aligned} \text{ii) } P(X < 6) &= P(X=0) + P(X=1) + P(X=3) + P(X=4) + P(X=5) \\ &= 0 + K + 2K + 2K + 3K \\ &= 8K \\ &= 8(0.102) \\ &= 0.816 \end{aligned}$$

$$\begin{aligned} P(X \geq 6) &= P(X=6) + P(X=7) = 1 - P(X < 6) \\ &= 1 - 0.816 \\ &= 0.184 \end{aligned}$$

$$\begin{aligned}
 \text{iii) } P(0 < X < 5) &= P(X=1) + P(X=3) + P(X=4) \\
 &= K + 2K + 2K \\
 &= 5K \\
 &= 5(0.102) \\
 &= 0.51
 \end{aligned}$$

★ 2) A R.V 'X' has the following probability distribution

$X = x_i$	-2	-1	0	1	2	3
$P(X = x_i)$	0.1	K	0.2	2K	0.3	K

Find,

i) K

ii) mean

iii) variance

iv) standard deviation

Sol. i) We know that $\sum P(X = x_i) = 1$

$$\Rightarrow 0.1 + K + 0.2 + 2K + 0.3 + K = 1$$

$$\Rightarrow \boxed{K = 0.1}$$

ii) mean $\mu = E(X) = \sum x_i \cdot P(X = x_i)$

$$\Rightarrow \mu = E(X) = -2(0.1) - 1(K) + 0(0.2) + 1(2K) + 2(0.3) + 3K$$

$$= -0.2 - K + 0 + 2K + 0.6 + 3K$$

$$= 4K + 0.4$$

$$= 4(0.1) + 0.4$$

$$= 0.8$$

iii) Variance, $\sigma^2 = E(X^2) - [E(X)]^2$

$$\text{Now, } E(X^2) = \sum x_i^2 P(X = x_i)$$

$$\begin{aligned}
 &= (-2)^2(0.1) + (-1)^2(K) + 0^2(0.2) + 1^2(2K) \\
 &\quad + 2^2(0.3) + 3^2(K)
 \end{aligned}$$

$$= 0.4 + K + 0 + 2K + 1.2 + 9K$$

$$= 2.8$$

$$\text{Variance } \sigma^2 = E(X^2) - [E(X)]^2$$

$$= 2.8 - (0.8)^2$$

$$\boxed{\sigma^2 = 2.16}$$

iv) Standard deviation

$$\sigma = \sqrt{2.16} = 1.469$$

★ Problems on Probability density [continuous probability distribution]

i) If a R.V 'x' has the probability density $f(x)$ as

$$f(x) = \begin{cases} 2e^{-2x} & , \text{ for } x > 0 \\ 0 & , \text{ for } x \leq 0 \end{cases}$$

Find the probabilities that it will take a value

i) blw 1 and 3

ii) Greater than 0.5

Sol.

$$\begin{aligned} \text{i) } P(1 < X < 3) &= \int_1^3 f(x) dx \\ &= \int_1^3 2e^{-2x} dx \quad (x > 0) \\ &= 2 \left[\frac{e^{-2x}}{-2} \right]_1^3 \\ &= -[e^{-6} - e^{-2}] \\ &= e^{-2} - e^{-6} \\ &= \frac{1}{e^2} - \frac{1}{e^6} \end{aligned}$$

$$\begin{aligned}
 \text{ii) } P(X > 0.5) &= \int_{0.5}^{\infty} f(x) dx \\
 &= \int_{0.5}^{\infty} 2e^{-2x} dx \quad (x > 0) \\
 &= 2 \left[-\frac{e^{-2x}}{2} \right]_{0.5}^{\infty} \\
 &= -[e^{-\infty} - e^{-1}] \\
 &= -[0 - \frac{1}{e}] \\
 &= \frac{1}{e}
 \end{aligned}$$

$$\left\{ \begin{array}{l} e^{-\infty} = 0 \\ e^{\infty} = \infty \end{array} \right.$$

2) If the probability density of a R.V is given by

$$f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \rightarrow f(x) = 0 \end{cases}$$

Find the value of k , and the probability that a R.V having this probability density will take on a value

- i) between 0.1 and 0.2
- ii) greater than 0.5

Sol. We know that,

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_{-\infty}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^{\infty} f(x) dx = 1$$

$$\Rightarrow 0 + \int_0^1 k(1-x^2) dx + 0 = 1$$

$$\Rightarrow k \left[x - \frac{x^3}{3} \right]_0^1 = 1$$

$$\Rightarrow k \left[\left(1 - \frac{1}{3}\right) - 0 \right] = 1$$

$$K \left[\frac{2}{3} \right] = 1$$

$$\Rightarrow \boxed{K = \frac{3}{2}}$$

$$\therefore f(x) = \begin{cases} \frac{3}{2}(1-x^2) & , 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$i) P(0.1 < x < 0.2) = \int_{0.1}^{0.2} f(x) dx$$

$$= \int_{0.1}^{0.2} \frac{3}{2}(1-x^2) dx$$

$$= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_{0.1}^{0.2}$$

$$= \frac{3}{2} \left[\left(0.2 - \frac{(0.2)^3}{3} \right) - \left(0.1 - \frac{(0.1)^3}{3} \right) \right]$$

$$= 0.147$$

$$ii) P(x > 0.5) = \int_{0.5}^{\infty} f(x) dx$$

$$= \int_{0.5}^1 f(x) dx + \int_1^{\infty} f(x) dx$$

$$= \int_{0.5}^1 \frac{3}{2}(1-x^2) dx + 0$$

$$= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_{0.5}^1$$

$$= \frac{3}{2} \left[\left(1 - \frac{1}{3} \right) - \left(0.5 - \frac{(0.5)^3}{3} \right) \right]$$

$$= 0.3125$$

3) The probability density $f(x)$ of a continuous random variable is given by $f(x) = ce^{-|x|}$, $-\infty < x < \infty$
 show that $c = \frac{1}{2}$ and find the mean and variance of the distribution, also find the probability that variate lies between 0 and 4 $\Rightarrow P(0 < x < 4) = ?$

Sol. i) We know that,

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

this implies to $f(x)$

$$\int_{-\infty}^{\infty} ce^{-|x|} dx = 1$$

$$\Rightarrow c \cdot \int_{-\infty}^{\infty} e^{-|x|} dx = 1$$

$$\Rightarrow c \cdot 2 \int_0^{\infty} e^{-|x|} dx = 1 \quad (\because \text{Since integrand is even})$$

$$\Rightarrow 2c \int_0^{\infty} e^{-x} dx = 1 \quad (\because 0 < x < \infty \Rightarrow |x| = x)$$

$$\Rightarrow 2c \left[\frac{e^{-x}}{-1} \right]_0^{\infty} = 1$$

$$\Rightarrow -2c [e^{-\infty} - e^0] = 1$$

$$\Rightarrow -2c [0 - 1] = 1$$

$$\Rightarrow 2c = 1$$

$$\boxed{c = \frac{1}{2}}$$

$$\left\{ \begin{array}{l} f(-x) = f(x), \text{ even} \\ f(-x) = -f(x), \text{ odd} \end{array} \right.$$

$$f(x) = e^{-|x|}$$

$$f(x) = e^{-|-x|} = e^{-|x|} = f(x)$$

$f(x)$ is even

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

$$\left\{ \begin{array}{l} e^{-\infty} = 0 \\ e^{\infty} = \infty \end{array} \right.$$

ii) Mean:

$$\mu = E(x) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x \cdot e^{-|x|} dx$$

$$\boxed{\mu = E(x) = 0} \quad (\because \text{Since integrand is odd})$$

iii) Variance:

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= E(x^2) - (0)^2$$

$$[\because \mu = E(x) = 0]$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f(x) dx$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 \cdot e^{-|x|} dx$$

$$= \frac{1}{2} \cdot 2 \int_0^{\infty} x^2 e^{-|x|} dx$$

$[\because \text{Since integrand is even}]$

$$= \int_0^{\infty} x^2 \cdot e^{-|x|} dx$$

$$= \int_0^{\infty} x^2 \cdot e^{-x} dx$$

$$[0 < x < \infty \Rightarrow |x| = x]$$

$$= \left[x^2 \left(\frac{e^{-x}}{-1} \right) - 2x(e^{-x}) + 2 \left(\frac{e^{-x}}{-1} \right) \right]_0^{\infty}$$

$$= \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^{\infty}$$

$$= 0 - (-2e^0)$$

$$\boxed{\sigma^2 = 2}$$

$$\{e^{-\infty} = 0\}$$

$$\text{iv) } P(0 < x < 4)$$

$$P(0 < x < 4) = \int_0^4 f(x) dx$$

$$= \int_0^4 \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2} \int_0^4 e^{-|x|} dx$$

$$= \frac{1}{2} \int_0^4 e^{-x} dx$$

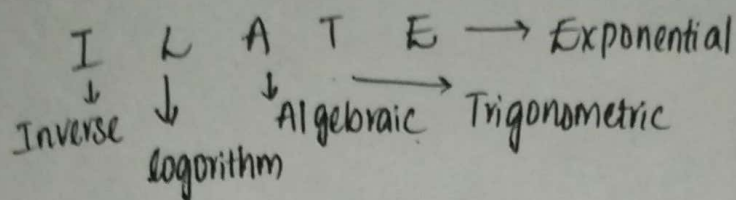
$$\left[\begin{array}{l} 0 < x < 4 \\ |x| = x \end{array} \right]$$

$$= \frac{1}{2} \left(\frac{e^{-x}}{-1} \right)_0^4$$

$$= \frac{1}{2} [e^{-4} - e^0]$$

$$= \frac{1}{2} \left[1 - \frac{1}{e^4} \right]$$

• Order



$$\Rightarrow \int uv = u \int v - \int [u' \int v]$$

$$\Rightarrow u' = \frac{du}{dx}$$

$$\bullet \int x \cdot \sin x \, dx = x(-\cos x) - 1 \cdot (-\sin x) + 0$$

$$\bullet \int x^2 \sin x \, dx = x^2(-\cos x) - 2x(-\sin x) + 2(\cos x) - 0$$

$$\bullet \int x \cos x \, dx = x(\sin x) - 1 \cdot (\cos x) + 0$$

$$\bullet \int x^2 \cos x \, dx = x^2(\sin x) - 2x(\cos x) + 2(0)$$

$$\Rightarrow \int uv = u \int v$$

Integration of u and v is derivation of first term and integration of second term independently with the alternate signs.

4) For the Continuous probability $f(x) = kx^2 e^{-x}$ when $x \geq 0$, Find i) K ii) Mean iii) Variance.

Sol. We know that,

$$\Rightarrow \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx = 1$$

$$\Rightarrow 0 + \int_0^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^{\infty} kx^2 e^{-x} dx = 1$$

$$\Rightarrow k \int_0^{\infty} \underbrace{x^2}_u \underbrace{e^{-x}}_v dx = 1 \quad uv' + uv''$$

$$\Rightarrow \int uv dx = u \int v dx - \int \frac{d}{dx}(u) \int v dx dx$$

$$\Rightarrow k [-x^2 e^{-x} + \int 2x \cdot e^{-x} dx] = 1$$

$$\Rightarrow k [-x^2 e^{-x} + 2 \int \underbrace{x}_u \underbrace{e^{-x}}_v dx] = 1$$

$$\Rightarrow k [-x^2 e^{-x} + 2 [-x e^{-x} - \int 1 \cdot e^{-x} dx]] = 1$$

$$\Rightarrow k [-x^2 e^{-x} - 2x e^{-x} + 2(-e^{-x})]_0^{\infty} = 1$$

$$\Rightarrow k [0 + 2e^0] = 1$$

$$2k = 1$$

$$\boxed{k = 1/2}$$

ii) Mean:

$$\mu = \int_{-\infty}^{\infty} x f(x) dx \quad x \geq 0$$

$$\mu = \int_0^{\infty} x \cdot kx^2 e^{-x} dx$$

$$= \int_0^{\infty} x \cdot \frac{1}{2} x^2 e^{-x} dx$$

$$= \frac{1}{2} \int_0^{\infty} x^3 e^{-x} dx$$

$$= \frac{1}{2} \int_0^{\infty} -x^3 e^{-x} + \int \frac{d}{dx} (x^3) \int e^{-x} dx dx$$

$$= \frac{1}{2} \int_0^{\infty} -x^3 e^{-x} + \int_{uv} 3x^2 e^{-x} dx$$

$$M = \frac{1}{2} [-x^3 e^{-x} + 3 \int -x^2 e^{-x} + \int 2x e^{-x} dx]$$

$$= \frac{1}{2} [-x^3 e^{-x} - 3x^2 e^{-x} + 6 \int_{uv} x e^{-x} dx]$$

$$= \frac{1}{2} [-x^3 e^{-x} - 3x^2 e^{-x} + 6 [-x e^{-x} + \int e^{-x} dx]]$$

$$= \frac{1}{2} [-x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x}]_0^{\infty}$$

$$= \frac{1}{2} [0 + 6e^0]$$

$$= \frac{1}{2} (6)$$

$$\boxed{M = 3}$$

iii) Variance :

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= E(x)^2 - [E(x)]^2$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} x^2 e^{-x} dx - (3)^2$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^4 e^{-x} dx - 9$$

$$\int uv dx = u \int v dx - \int \frac{d}{dx} (u) \int v dx dx$$

$$= \frac{1}{2} [-x^4 e^{-x} + \int 4x^3 e^{-x} dx] - 9$$

$$= \frac{1}{2} [-x^4 e^{-x} + 4 [-x^3 e^{-x} + \int 3x^2 e^{-x} dx]] - 9$$

$$= \frac{1}{2} [-x^4 e^{-x} - 4x^3 e^{-x} + 12 \int x^2 e^{-x} dx] - 9$$

$$= \frac{1}{2} [-x^4 e^{-x} - 4x^3 e^{-x} + 12 [-x^2 e^{-x} + \int 2x e^{-x} dx] dx] - 9$$

$$= \frac{1}{2} [-x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} + 24 \int x e^{-x} dx] - 9$$

$$= \frac{1}{2} [-x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} + 24 [-x e^{-x} + \int 1 \cdot e^{-x} dx] dx] - 9$$

$$= \frac{1}{2} [-x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24x e^{-x} + 24e^{-x}]_0^{\infty} - 9$$

$$= \frac{1}{2} [0 + 24e^{-x}]_0^{\infty} - 9$$

$$= \frac{1}{2} [24e^0] - 9$$

$$= \frac{1}{2} (24 \times 1) - 9$$

$$= 12 - 9$$

$$\boxed{\sigma^2 = 3}$$

5) Suppose a continuous R.V 'x' has the probability density

$f(x) = k(1-x^2)$ for $0 < x < 1$ and $f(x) = 0$ otherwise. Find

i) k ii) Mean iii) Variance.

Sol.

We know that,

$$\int_{-\infty}^{\infty} f(x) dx$$

$$\Rightarrow \int_0^1 k(1-x^2) dx = 1$$

$$\Rightarrow k \int_0^1 (1-x^2) dx = 1$$

$$\Rightarrow k \left(x - \frac{x^3}{3} \right)_0^1$$

$$\Rightarrow K \left[1 - \frac{1}{3} \right] = 1$$

$$\Rightarrow K \left[\frac{3-1}{3} \right] = 1$$

$$\Rightarrow K \left[\frac{2}{3} \right] = 1$$

$$\boxed{K = 3/2}$$

ii) mean:

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^1 x K(1-x^2) dx$$

$$= K \int_0^1 x(1-x^2) dx$$

$$= K \int_0^1 (x - x^3) dx$$

$$= K \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1$$

$$= K \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= K \left[\frac{1}{4} \right]$$

$$= \frac{3}{2} \left[\frac{1}{4} \right]$$

$$= \frac{3}{8}$$

$$\boxed{\mu = \frac{3}{8}}$$

iii) Variance:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - 4^2$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 \cdot \frac{3}{2} dx - \frac{9}{64}$$

$$\Rightarrow \frac{3}{2} \int_{-\infty}^{\infty} x^2 dx - \frac{9}{64}$$

$$\Rightarrow \frac{3}{2} \int_0^1 x^2 dx - \frac{9}{64}$$

$$\Rightarrow \frac{3}{2} \left(\frac{x^3}{3} \right)'_0 - \frac{9}{64} \Rightarrow \frac{3}{2} \left(\frac{1}{3} \right) - \frac{9}{64} \Rightarrow \frac{1}{2} - \frac{9}{64}$$

(or)

$$\int_{-\infty}^{\infty} x^2 f(x) dx$$

$$\Rightarrow K \int_0^1 x^2 (1-x^2) dx$$

$$\Rightarrow K \int_0^1 (x^2 - x^4) dx$$

$$\Rightarrow K \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$\Rightarrow K \left[\frac{1}{3} - \frac{1}{5} \right]$$

$$\Rightarrow \frac{3}{2} \left[\frac{5-3}{15} \right] \Rightarrow \frac{3}{2} \left[\frac{2}{15} \right]$$

$$\sigma^2 = \frac{1}{5} - \left(\frac{3}{8} \right)^2$$

$$= \frac{1}{5} - \frac{9}{64}$$

$$= 0.2 - 0.1406$$

$$\sigma^2 = 0.666.$$

Note:

- 1) Binomial
- 2) Poisson
- 3) Normal

- Binomial & Poisson distributions are modeled for discrete R.V and normal distribution is modeled for continuous R.V

⇒ Binomial Distribution

Binomial Distribution of a discrete R.V 'X' is denoted by

$$B(x; n, p) \text{ (or) } P(X=x) \text{ (or) } p(x)$$

and is defined by

$$B(x; n, p) = P(X=x) = p(x) = {}^n C_x p^x q^{n-x}$$

$$x = 0, 1, 2, \dots, n$$

Here n is no. of trials

p is probability of success in each trial

q is probability of failure.

where $x = 0, 1, 2, \dots, n$ are no. of success

Note

1) Binomial distribution is modeled for Discrete random variable.

2) In a B.D, $n = \max. 40$

3) $p + q = 1$ i.e., probability of success + probability of failure

★ 4) ~~n, q~~ n, p are called parameters of a B.D

5) In a B.D. ' n ' is very small and ' p ' is very large.

> Mean of B.D

Mean of B.D is defined by

$$\mu = E(x) = np$$

> Variance of B.D

Variance of B.D is defined by

$$\sigma^2 = npq$$

> Note:

- The standard equation of B.D is $\sigma = \sqrt{npq}$

* Ex 1: Show that mean of B.D is np

Sol: Mean of a Discrete R.V 'x' is given by

$$\mu = E(x) = \sum_{x=0}^n x \cdot P(X=x)$$

B.D is given by,

$$P(X=x) = p(x) = {}^n C_x p^x q^{n-x}, \quad x=0, 1, \dots, n$$

$$\therefore \mu = E(x) = \sum_{x=0}^n x \cdot p(x)$$

$$= 0 + 1p(1) + 2 \cdot p(2) + \dots + n(p(n))$$

$$= nC_1 p^1 q^{n-1} + 2nC_2 p^2 q^{n-2} + \dots + n \cdot nC_n p^n q^0$$

$$= npq^{n-1} + \frac{2(n(n-1))}{2!} p^2 q^{n-2} + \dots + n \cdot 1 \cdot p^n$$

$$(x+y)^n = nC_0 y^n +$$

$$nC_1 x^1 y^{n-1} + \dots = np [q^{n-1} + (n-1)pq^{n-2} + \dots + p^{n-1}]$$

$$= np(p+q)^{n-1}$$

$$= np(1)^{n-1}$$

$$= np$$

$$[\because p+q=1]$$

$$\therefore \boxed{\mu = E(x) = np}$$

2) Show that variance of B.D is npq

B.D is given by

$$P(X=x) = p(x) = nC_x p^x q^{n-x}, \quad x=0,1,2,\dots,n$$

$\therefore$ Variance of discrete R.V 'x' is given by

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= \sum_{x=0}^n x^2 \cdot p(x) - (np)^2$$

$$= \sum_{x=0}^n x^2 (p(x)) - (np)^2$$

[take x^2 as $x(x-1)+x$]

$$= \sum_{x=0}^n [x(x-1)+x] p(x) - (np)^2$$

$$= \sum_{x=0}^n x(x-1) p(x) + \sum_{x=0}^n x \cdot p(x) - (np)^2$$

$$= \sum_{x=0}^n x(x-1) p(x) + n - (np)^2$$

$$= \sum_{x=0}^n x(x-1) p(x) + np - (np)^2$$

$$= 0+0+2 \cdot 1 \cdot p(2) + 3 \cdot 2 \cdot p(3) + \dots + n(n-1) p(n) + np - (np)^2$$

$$= 2nC_2 p^2 q^{n-2} + 3 \cdot 2 \cdot nC_3 p^3 q^{n-3} + \dots + n(n-1)nC_n p^n + np - (np)^2$$

$$= \frac{2 \cdot n(n-1)}{2!} p^2 q^{n-2} + \frac{3 \cdot 2 \cdot n(n-1)(n-2)}{3!} p^3 q^{n-3} + \dots + n(n-1) p^n + np - (np)^2$$

$$= n(n-1)p^2 [q^{n-2} + (n-2)pq^{n-3} + \dots + p^{n-2}] + np - (np)^2$$

$$= n(n-1)p^2 (p+q)^{n-2} + np - (np)^2$$

$$= n(n-1)p^2 (1)^{n-2} + np - (np)^2$$

$$= n^2 p^2 - np^2 + np - n^2 p^2$$

$$= np(1-p)$$

$$\sigma^2 = npq$$

$$\left[\because p+q=1 \Rightarrow 1-p=q \right]$$

$\Rightarrow$ Relation blw Mean and Variance of B.D

$$\text{Mean of B.D} = np$$

$$\text{Variance of B.D} = npq$$

$$M = np \quad \text{--- (1)}$$

$$\sigma^2 = npq \quad \text{--- (2)}$$

$$(p+q=1 \Rightarrow q < 1)$$

$$\frac{\text{mean}}{\text{variance}} = \frac{np}{npq} = \frac{1}{q} > 1$$

$$\star \boxed{\text{mean} > \text{variance}}$$

i.e., In B.D, the mean is always greater than Variance.

3) A coin is tossed 6 tyms. Find the probability of getting 4 heads.

Sol: In a single toss of a coin,

$$\text{let } p = \text{probability of getting a head} = \frac{1}{2} \text{ (success)}$$

$$\therefore q = 1-p \Rightarrow 1 - \frac{1}{2} \Rightarrow \frac{1}{2}$$

$$\text{Here, } n=6$$

$$p = \frac{1}{2}$$

$$x = 4 \text{ (success)}$$

$$\text{B.D is given by } P(X=x) = {}^nC_x p^x q^{n-x}$$

$$P(X=4) = {}^6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2$$

$$= \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} \left(\frac{1}{2}\right)^6$$

$$= \frac{15}{64}$$

4) Determine the probability of getting the sum 6, exactly 3 times in 7 throws with a pair of fair dice.

Sol. In a single throw of a pair of dice, a sum 6 can occur in 5 ways

i.e., (1,5) (2,4) (3,3) (4,2) (5,1)

Let p = probability of occurrence of sum 6 in a single throw = $\frac{5}{36}$.

$$p = 5/36$$

$$q = 1 - p \Rightarrow 1 - \frac{5}{36}$$

$$q = \frac{31}{36}$$

Here, $n = 7$, $x = 3$ (success)

$\therefore$ B.D is given by

$$p(x=x) = {}^n C_x p^x q^{n-x}$$

$$p(x=3) = {}^7 C_3 \left(\frac{5}{36}\right)^3 \left(\frac{31}{36}\right)^4$$

$$= \frac{7 \times 6 \times 5}{1 \times 2 \times 3} \cdot \frac{5^3 \times 31^4}{(36)^7}$$

$$= 0.0515$$

{ over $\rightarrow >$
under $\rightarrow <$ }

{ at least $\rightarrow \geq$
at most $\rightarrow \leq$ }

5) A Die is thrown 6 times. If getting an even number is on success. Find the probability of

i) atleast 1 success

ii) ≤ 3 success

iii) 4 success

Sol.

Here $n = 6$

i.e., 2, 4, 6.

In a single throw of a die, getting an even number is (2, 4, 6)

Let p = probability of getting an even number in a single

$$\text{throw} = \frac{3}{6} = \frac{1}{2}$$

$$p = \frac{1}{2}$$

$$q = 1 - p \Rightarrow 1 - \frac{1}{2} = \frac{1}{2}$$

B.D is given by

$$P(X=x) = {}^nC_x p^x q^{n-x}$$

$$i) P(X \geq 1) = P(X=1) + P(X=2) + \dots + P(X=6)$$

$$= 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - {}^6C_0 p^0 q^6$$

$$= 1 - 1 \cdot 1 \left(\frac{1}{2}\right)^6$$

$$= 1 - \frac{1}{64}$$

$$= \frac{63}{64}$$

$$\text{ii) } P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{1}{64} + {}^6C_1 p^1 q^5 + {}^6C_2 p^2 q^4 + {}^6C_3 p^3 q^3$$

$$= \frac{1}{64} + 6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^5 + 15\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 + 20\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3$$

$$= \frac{1}{64} + \left(\frac{1}{2}\right)^6 [6 + 15 + 20]$$

$$= \frac{21}{32}$$

$$\text{iii) } P(X=4) = {}^6C_4 p^4 q^2$$

$$= {}^6C_2 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2$$

$$= \frac{15}{64}$$

6) 10 coins are thrown simultaneously. Find the probability of getting atleast 7 heads.

Sol. Here $n=10$.

In a single throw of a coin,

let p = probability of getting a head = $\frac{1}{2}$

$$p = \frac{1}{2}$$

$$q = 1 - p \Rightarrow 1 - \frac{1}{2}$$

$$q = \frac{1}{2}$$

$$x = 7$$

B.D is given by

$$P(X=x) = {}^nC_x p^x q^{n-x}$$

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= {}^{10}C_7 p^7 q^3 + {}^{10}C_8 p^8 q^2 + {}^{10}C_9 p^9 q$$

$$+ {}^{10}C_{10} p^{10}$$

$$\begin{aligned}
 &= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \\
 &= \left(\frac{1}{2}\right)^{10} [{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}] \\
 &= 0.1719.
 \end{aligned}$$

7) If 3 of 20 tyres are defective and 4 of them are randomly chosen for inspection. What is the probability that only one of the defective tyre will be included.

Sol. let p = probability of defective tyre

$$p = \frac{3}{20}$$

$$q = 1 - p$$

$$q = 1 - \frac{3}{20} \Rightarrow q = \frac{17}{20}$$

here $n = 4$

B.D is given by

$$P(X=x) = {}^nC_x p^x q^{n-x}$$

$$P(X=1) = P(X=1)$$

$$= {}^4C_1 p q^3$$

$$= {}^4C_1 \left(\frac{3}{20}\right) \left(\frac{17}{20}\right)^3$$

$$= 0.3684$$

8) Determine the B.D for the ~~mean~~ which mean is 4 and variance is 3.

Sol. Mean & Variance of B.D is given by

$$\mu = np = 4 \quad \text{--- (1)}$$

$$\sigma^2 = npq = 3 \quad \text{--- (2)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \boxed{q = \frac{3}{4}}$$

$$p = 1 - q$$

$$= 1 - \frac{3}{4}$$

$$\boxed{p = \frac{1}{4}}$$

$$\textcircled{1} \Rightarrow n \times \frac{1}{4} = 4 \Rightarrow \boxed{n = 16}$$

$$\begin{aligned} \text{B.D is } &= (p+q)^n = \left(\frac{1}{4} + \frac{3}{4}\right)^6 \\ &= 1 \end{aligned}$$

★ 9) If Out of 800 families with 5 children, how many would you expect to have

i) 3 boys

ii) 5 girls

iii) Either 2 (or) 3 boys.

Sol. Assume that,

Equal probabilities for boys and girls

★ here $n=5$

Let p = probability of a boy (success) = $\frac{1}{2}$

$$q = 1 - p$$

$$= 1 - \frac{1}{2}$$

$$q = \frac{1}{2}$$

B.D is given by

$$P(X=x) = {}^nC_x p^x q^{n-x}$$

i) $P(3 \text{ boys})$

$$\Rightarrow P(X=3) = {}^5C_3 p^3 q^{5-3}$$

$$= {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$\Rightarrow$ 800 No. of families having 3 boys =

$$800 \times \frac{10}{32}$$

$$= 250.$$

250 families have 3 boys & 2 girls

$$\text{ii) } P(5 \text{ girls}) = P(0 \text{ boys}) = P(X=0)$$

B.D is given by,

$${}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5$$

$$= 1 \left(\frac{1}{2}\right)^5$$

$$= \frac{1}{32}$$

$$\therefore \text{No. of families having 5 girls} \Rightarrow 800 \times \frac{1}{32} \\ = 25.$$

$$\text{iii) } P(\text{either 2 or 3 boys})$$

$$\Rightarrow P(X=2) + P(X=3)$$

$$\Rightarrow {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \left(\frac{1}{2}\right)^5 [{}^5C_2 + {}^5C_3]$$

$$\Rightarrow \frac{1}{32} \times (10+10)$$

$$\Rightarrow \frac{20}{32}$$

$$\therefore \text{No. of families having either 2 or 3 boys} \Rightarrow$$

$$800 \times \frac{20}{32}$$

$$\Rightarrow 500.$$

10) Assume that 50% of all engineering students are good in mathematics. Determine the probability that among 18 engineering students

i) Exactly 10

ii) Atleast 10

iii) Atmost 8

iv) Atleast 2 and atmost 9, are good in mathematics

Sol. Here $n=18$

Let p = probability of success = 50% = $\frac{50}{100} = \frac{1}{2}$

$$q = 1 - p \Rightarrow 1 - \frac{1}{2} \Rightarrow \frac{1}{2}$$

i) $P(X=x)$

$$P(X=10)$$

B.D is given by

$$P(X=10) = {}^{18}C_{10} p^{10} q^{18-10}$$

$$= {}^{18}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^8 \Rightarrow {}^{18}C_{10} \left(\frac{1}{2}\right)^{18}$$

$$= 0.1669.$$

ii) atleast 10.

$$P(X \geq 10)$$

B.D is given by

$$P(X \geq 10) = P(X=10) + P(X=11) + P(X=12) + P(X=13) +$$

$$P(X=14) + P(X=15) + P(X=16) + P(X=17) + P(X=18)$$

$$= {}^{18}C_{10} p^{10} q^{18-10} + {}^{18}C_{11} p^{11} q^7 + {}^{18}C_{12} p^{12} q^6 + {}^{18}C_{13} p^{13} q^5$$

$$+ {}^{18}C_{14} p^{14} q^4 + {}^{18}C_{15} p^{15} q^3 + {}^{18}C_{16} p^{16} q^2 + {}^{18}C_{17} p^{17} q^1$$

$$+ {}^{18}C_{18} p^{18} q^0$$

$$= {}^{18}C_{10} \left(\frac{1}{2}\right)^{18} + {}^{18}C_{11} \left(\frac{1}{2}\right)^{18} + {}^{18}C_{12} \left(\frac{1}{2}\right)^{18} + {}^{18}C_{13} \left(\frac{1}{2}\right)^{18}$$

$$+ 18C_{14}\left(\frac{1}{2}\right)^{18} + 18C_{15}\left(\frac{1}{2}\right)^{18} + 18C_{16}\left(\frac{1}{2}\right)^{18} + 18C_{17}\left(\frac{1}{2}\right)^{18} + 18C_{18}\left(\frac{1}{2}\right)^{18}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{18} [18C_{10} + 18C_{11} + 18C_{12} + 18C_{13} + 18C_{14} + 18C_{15} + 18C_{16} + 18C_{17} + 18C_{18}]$$

$$\Rightarrow 0.4072$$

iii) atmost 8

$$P(X \leq 8) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$$

$$= 18C_0 p^0 q^{18} + 18C_1 p^1 q^{17} + 18C_2 p^2 q^{16} + 18C_3 p^3 q^{15} + 18C_4 p^4 q^{14} + 18C_5 p^5 q^{13} + 18C_6 p^6 q^{12} + 18C_7 p^7 q^{11} + 18C_8 p^8 q^{10}$$

$$= 18C_0 \left(\frac{1}{2}\right)^{18} + 18C_1 \left(\frac{1}{2}\right)^{18} + 18C_2 \left(\frac{1}{2}\right)^{18} + 18C_3 \left(\frac{1}{2}\right)^{18} + 18C_4 \left(\frac{1}{2}\right)^{18} + 18C_5 \left(\frac{1}{2}\right)^{18} + 18C_6 \left(\frac{1}{2}\right)^{18} + 18C_7 \left(\frac{1}{2}\right)^{18} + 18C_8 \left(\frac{1}{2}\right)^{18}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{18} [18C_0 + 18C_1 + 18C_2 + 18C_3 + 18C_4 + 18C_5 + 18C_6 + 18C_7 + 18C_8]$$

$$\Rightarrow 0.4072$$

iv) $P(2 \leq X \leq 9)$

$$P(2 \leq X \leq 9) = P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9)$$

$$= 18C_2 \left(\frac{1}{2}\right)^{18} + 18C_3 \left(\frac{1}{2}\right)^{18} + 18C_4 \left(\frac{1}{2}\right)^{18} + 18C_5 \left(\frac{1}{2}\right)^{18} + 18C_6 \left(\frac{1}{2}\right)^{18} + 18C_7 \left(\frac{1}{2}\right)^{18} + 18C_8 \left(\frac{1}{2}\right)^{18} + 18C_9 \left(\frac{1}{2}\right)^{18}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{18} [18C_2 + 18C_3 + 18C_4 + 18C_5 + 18C_6 + 18C_7 + 18C_8 + 18C_9]$$

$$\Rightarrow 0.5926$$

► Poisson Distribution (P.D):

Note: P.D of a discrete R.V 'x' is denoted by

$$P(X=x) \text{ (or) } P(x)$$

and is defined by,

$$P(X=x) = P(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} \quad x = 0, 1, 2, \dots$$

here $\lambda > 0$ is called parameter of the distribution.

Note:

1) In P.D, n is very large and p is small

$$2) \lambda = np$$

eg 1) No. of accidents on a highway (n = no. of vehicles and p = prob. of accidents).

eg 2) No. of defectives in a production center.

Mean & Variance of P.D

Mean & Variance of P.D is defined as

$$\text{mean, } \mu = E(X) = \lambda$$

$$\text{variance, } \sigma^2 = \lambda$$

Note:

1) Mean & Variance of P.D are same i.e., λ

2) Standard deviation of P.D = $\sqrt{\lambda}$

Problems

n is very large
&
 p is very small

i) Show that mean & variance of P.D is λ

Sol. P.D is defined as

$$p(X=x) = p(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, \quad x=0, 1, 2, \dots$$

i) mean of Discrete R.V 'x' is given by

$$\mu = E(x) = \sum_{x=0}^{\infty} x \cdot p(x=x)$$

$$= \sum_{x=0}^{\infty} x \cdot p(x)$$

$$= 0 + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + \dots$$

$$= p(1) + 2p(2) + 3p(3) + \dots$$

$$= \frac{e^{-\lambda} \cdot \lambda}{1!} + \frac{2e^{-\lambda} \cdot \lambda^2}{2!} + \frac{3e^{-\lambda} \cdot \lambda^3}{3!} + \dots$$

$$= e^{-\lambda} \cdot \lambda \left[1 + \lambda + \frac{\lambda^2}{2} + \dots \right]$$

$$= e^{-\lambda} \cdot \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right]$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$\boxed{\mu = E(x) = \lambda}$$

ii) Variance

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= \sum_{x=0}^{\infty} x^2 p(x=x) = (\lambda)^2 \quad \left[\because E(x) = \lambda \right]$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] p(x) - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) p(x) + \sum_{x=0}^{\infty} x \cdot p(x) - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot p(x) + \mu - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot p(x) + \lambda - \lambda^2 \quad [\because \mu = E(x) = \lambda]$$

$$= [0 + 0 + 2 \cdot 1 \cdot p(2) + 3 \cdot 2 p(3) + \dots] + \lambda - \lambda^2$$

$$= [2 p(2) + 3 \cdot 2 p(3) + \dots] + \lambda - \lambda^2$$

$$= \left[\frac{2 \cdot e^{-\lambda} \lambda^2}{2!} + 3 \cdot 2 \cdot \frac{e^{-\lambda} \lambda^3}{3!} + \dots \right] + \lambda - \lambda^2 \quad [\because \textcircled{1}]$$

$$= e^{-\lambda} \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots \right] + \lambda - \lambda^2$$

$$= e^{-\lambda} \cdot \lambda^2 e^{\lambda} + \lambda - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\sigma^2 = \lambda}$$

Hence proved.

2) If the probability that the individual suffer of a bad reaction from an infection is 0.001. Determine the probability that out of 2000 individuals

i) exactly 3

ii) more than 2

iii) none

iv) more than 1 individual suffer a bad reaction.

Sol. here $n = 2000$ very large
 $p = 0.001$ very small

P.D is given by,

$$p(x=x) = p(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x=0, 1, 2, \dots$$

$$\lambda = np = 2000 \times 0.001$$

$$\boxed{\lambda = 2}$$

$$i) P(X=3) = \frac{e^{-\lambda} \lambda^3}{3!} \Rightarrow \frac{e^{-2} \cdot 2^3}{6} \Rightarrow \frac{8}{6e^2} = 0.1804$$

$$ii) P(X > 2) = P(X=3) + P(X=4) + \dots + P(X=2000) \\ = 1 - P(X \leq 2)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[\frac{e^{-\lambda} \lambda^0}{0!} + \frac{e^{-\lambda} \lambda^1}{1!} + \frac{e^{-\lambda} \lambda^2}{2!} \right]$$

$$= 1 - e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2} \right] \quad [0! = 1]$$

$$= 1 - e^{-2} \left[1 + 2 + \frac{4}{2} \right]$$

$$= 0.3233$$

$$iii) P(X=0) = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda} = e^{-2} = \frac{1}{e^2} = 0.1353$$

$$iv) P(X > 1) = P(X=2) + P(X=3) + P(X=4) + \dots + P(X=2000)$$

$$= 1 - P(X \leq 1)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\frac{e^{-\lambda} \lambda^0}{0!} + \frac{e^{-\lambda} \lambda^1}{1!} \right]$$

$$= 1 - e^{-\lambda} [1 + \lambda]$$

$$= 1 - e^{-2} [1 + 2]$$

$$= 0.594$$

3) 2% of the items of a factory are defective. The items are packed in boxes. What is the probability that there are

i) 2% of the defective items

ii) at least 3 defective items

iii) at least 3 defective items in a box of 100 items.

Sol here, $n = 100$

Let p = prob. of defective items

$$= 2\% = \frac{2}{100} = 0.02$$

$$\lambda = np = 100 \times 0.02 = 2$$

$$\boxed{\lambda = 2}$$

P.D is given by

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x=0, 1, 2, 3, \dots$$

$$i) P(X=2) = \frac{e^{-\lambda} \lambda^2}{2!} = \frac{e^{-2} \cdot 2^2}{2} = \frac{4}{2e^2} = 0.2706$$

$$ii) P(X \geq 3) = P(X=3) + P(X=4) + \dots + P(X=100)$$

$$= 1 - P(X < 3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= \left[\frac{e^{-\lambda} \lambda^0}{0!} + \frac{e^{-\lambda} \lambda^1}{1!} + \frac{e^{-\lambda} \lambda^2}{2!} \right]$$

$$= 1 - e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2} \right] \quad \frac{\lambda^2}{2!}$$

$$= 1 - e^{-2} \left[1 + 2 + \frac{4}{2} \right]$$

$$= 0.3233$$

- 4) If a Bank received on average 6 bad cheques per day. What are the probability that it will receive 4 bad cheques on any given day.

Sol. Given $\lambda = 6$

Now To Find $P(X=4)$

P.D is given by

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, \quad x=0,1,2,\dots$$

$$P(X=4) = \frac{e^{-\lambda} \lambda^4}{4!}$$

$$= \frac{e^{-6} 6^4}{4!}$$

$$= 0.1339$$

- 5) A hospital switch board receives an average of 4 emergency calls in a 10 minute interval. What is the probability that

i) there are atmost 2 emergency calls in 10 min interval

ii) There are exactly 3 emergency calls in a 10 min interval

Sol here, $n=10$

$$p = \frac{4}{10} = 0.4$$

$$\lambda = np = 10 \times 0.4 = 4$$

$$P(X=x) \Rightarrow \frac{e^{-\lambda} \lambda^x}{x!} \Rightarrow \frac{e^{-4} \cdot 4^x}{x!}$$

$$i) P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-4} 4^0}{0!} + \frac{e^{-4} 4^1}{1!} + \frac{e^{-4} 4^2}{2!}$$

$$= \frac{0.01831 \times 1}{1} + \frac{0.01831 \times 4}{1} + \frac{0.01831 \times 16}{2}$$

$$= 0.01831 + 0.07324 + 0.1464$$

$$= 0.2379$$

$$ii) P(X=3) = \frac{e^{-4} 4^3}{3!} = \frac{0.01831 \times 64}{6}$$

$$= 0.1953$$

6) If a R.V has a poisson distribution such that

$$P(1) = P(2) \text{ Find}$$

i) Mean of the distribution

ii) $P(4)$

iii) $P(X \geq 1)$

iv) $P(1 < X < 4)$

Sol $P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, x=0,1,2,\dots$

$$P(X=1) = P(X=2)$$

$$\frac{e^{-\lambda} \cdot \lambda^1}{1!} = \frac{e^{-\lambda} \cdot \lambda^2}{2!}$$

$$\lambda = \frac{\lambda^2}{2}$$

$$\lambda^2 = 2\lambda$$

$$2\lambda - \lambda^2 = 0$$

$$\lambda(2-\lambda) = 0$$

$$\boxed{\lambda = 2.0} \quad \boxed{\lambda = 2}$$

($\because \lambda > 0$)

$$i) P(X)=x = \frac{e^{-2} \cdot 2^x}{x!}$$

$$ii) P(X=4) = \frac{e^{-2} 2^4}{4!} \Rightarrow \frac{0.1353 \times 16}{24} \Rightarrow 0.0902$$

$$\begin{aligned} iii) P(X \geq 1) &= 1 - P(X \leq 1) \\ &= 1 - P(X=0) \\ &= 1 - \frac{e^{-2} 2^0}{0!} \\ &= 1 - \frac{0.1353 \times 1}{1} \\ &= 0.8647 \end{aligned}$$

$$\begin{aligned} iv) P(1 < X < 4) \\ &= P(X=1) + P(X=2) + P(X=3) + P(X=4) \\ &= \frac{e^{-2} 2^1}{1!} + \frac{e^{-2} 2^2}{2!} + \frac{e^{-2} 2^3}{3!} + \frac{e^{-2} 2^4}{4!} \\ &\Rightarrow 0.2706 + 0.2706 + 0.1804 + 0.0902 \\ &\Rightarrow 0.4511 \end{aligned}$$

7) Average no. of accidents on any day on a national highway is (1.8). Determine the probability that the no. of accidents are

i) at least 1 ≥ 1

ii) at most 1

Sol.

$$\lambda = 1.8$$

$$\begin{aligned} i) P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X=0) \end{aligned}$$

$$\text{by P.D} = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= 1 - \frac{e^{-1.8} (1.8)^0}{0!} \Rightarrow 1 - e^{-1.8}$$

$$P(X \geq 1) = 0.8347$$

$$ii) P(X \leq 1) = P(X=0) + P(X=1)$$

$$\lambda = 1.8$$

$$\Rightarrow \frac{e^{-1.8} (1.8)^0}{0!} + \frac{e^{-1.8} (1.8)^1}{1!}$$

$$\Rightarrow e^{-1.8} (1 + 1.8)$$

$$\Rightarrow e^{-1.8} (2.8)$$

$$\Rightarrow 0.4628.$$

8) If X is a poisson variate $P(X=0) = P(X=1)$,
Find $P(X=0)$ and using Find the probabilities at
 $x=1, 2, 3, 4, 5$

Sol. $P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$P(X=0) = P(X=1)$$

$$\frac{e^{-\lambda} \cdot \lambda^0}{0!} = \frac{e^{-\lambda} \cdot \lambda^1}{1!}$$

$$\frac{1}{1} = \lambda$$

$$\boxed{\lambda=1} \rightarrow \frac{e^{-1} (1)^x}{x!}$$

$$i) P(X=0) \Rightarrow \frac{e^{-1} 1^0}{0!}$$

$$= 0.3678$$

$$ii) P(X+1) = \frac{1}{x+1} P(X)$$

$$P(1) = 0.3678$$

$$P(2) = \frac{1}{1+1} \cdot P(1) = \frac{1}{2} (0.3678)$$

$$P(2) = 0.1839.$$

$$P(3) = \frac{1}{2+1} \cdot P(2) = \frac{1}{3} (0.1839)$$

$$P(3) = 0.0613$$

$$P(4) = \frac{1}{3+1} \cdot P(3) = \frac{1}{4} (0.0613)$$

$$P(4) = 0.015325$$

$$P(5) = \frac{1}{4+1} P(4) = \frac{1}{5} (0.015325)$$

$$P(5) = 0.003065$$

9) Suppose that 'x' has a poisson distribution of

$$P(X=2) = \frac{2}{3} (P=1). \text{ Find}$$

i) $P(X=0)$

ii) $P(X=3)$

Sol. $P(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$

$$P(X=2) = \frac{2}{3} (P(X=1))$$

$$\frac{e^{-\lambda} \cdot \lambda^2}{2!} = \frac{2}{3} \cdot \frac{e^{-\lambda} \cdot \lambda^1}{1!}$$

$$\frac{\lambda^2}{2} = \frac{2}{3} \cdot \lambda$$

$$3\lambda^2 = 4\lambda$$

$$3\lambda^2 - 4\lambda = 0$$

$$\lambda(3\lambda - 4) = 0$$

$$\lambda = 0$$

$$[\lambda > 0]$$

$$\boxed{\lambda = \frac{4}{3}}$$

$$i) P(X=0)$$

$$\frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-4/3} \cdot 1 = e^{-4/3}$$

$$\Rightarrow 0.2635$$

$$ii) P(X=3)$$

$$\frac{e^{-\lambda} \cdot \lambda^3}{3!} = \frac{e^{-4/3} \left(\frac{4}{3}\right)^3}{6}$$

$$= \frac{e^{-4/3} \cdot \frac{64}{27}}{6}$$

$$= \frac{64e^{-4/3}}{162}$$

$$P(X=3) = 0.10413.$$

10) If the Variance of a poisson distribution is 3 then Find the probability that,

$$i) X=0$$

$$ii) 0 < X \leq 3$$

$$iii) 1 \leq X < 4.$$

Sol. $P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$\boxed{\lambda = 3}$$

$$P(X=x) = \frac{e^{-3} \cdot 3^x}{x!}$$

$$i) P(X=0)$$

$$\frac{e^{-3} \cdot 3^0}{0!} \Rightarrow \frac{0.04978 \times 1}{1}$$

$$P(X=0) \Rightarrow 0.04978$$

$$ii) P(0 < X \leq 3) = P(1 \leq X < 4)$$

$$P(1) + P(2) + P(3)$$

$$\Rightarrow \frac{e^{-3} \cdot 3^1}{1!} + \frac{e^{-3} \cdot 3^2}{2!} + \frac{e^{-3} \cdot 3^3}{3!}$$

$$\Rightarrow \frac{0.04978 \times 3}{1} + \frac{0.04978}{2} + \frac{0.04978 \times 27}{6}$$

$$= 0.1493 + 0.2240 + 0.2240$$

$$= 0.5973$$

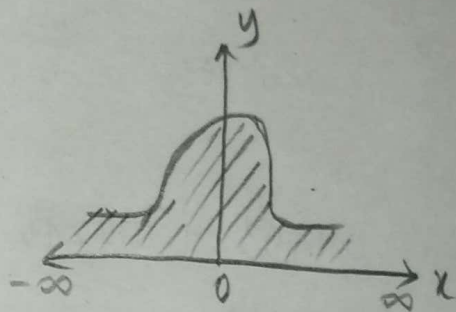
iii)
1 2 3

★ long ans.
Normal Distribution

A Continuous R.V 'x' is said to have Normal distribution with parameters mean (μ) and variance (σ^2) if probability density function $f(x)$ is given by.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\text{where, } z = \frac{x - \mu}{\sigma}$$

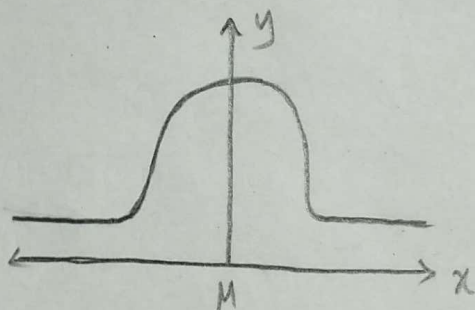


i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x) dx = 1$

total area under the normal curve above the x-axis is 1

> Properties of Normal Distribution.



- 1) N.D curve is bell-shaped
- 2) The N.D is Symmetrical about the mean (μ)
- 3) The mean, median and mode of the distribution coincide
- 4) Area under the Normal curve is 1
- 5) Normal curve is Asymptotic to the positive and negative axis.
- 6) Prob. that continuous R.V 'x' lies b/w x_1 & x_2 is given by

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x) dx = \int_{x_1}^{x_2} \frac{1}{\sigma - \sqrt{2\pi}} e^{-\frac{z^2}{2}} dx.$$

Standard Normal distribution

Normal Distribution with mean (μ) = 0 and Standard deviation (σ) = 1 is known as standard N.D.

- It is denoted by $N(0,1)$

Change of Scale from x -axis to z -axis (or) Convert Normal Distribution into Standard Normal Distribution.

To Convert N.D into Standard N.D ,

$$\text{put } z = \frac{x - \mu}{\sigma}$$

$$\Rightarrow \sigma z = x - \mu$$

$$\Rightarrow \sigma \cdot dz = dx - 0$$

$$\Rightarrow \sigma \cdot dz = dx$$

$$\text{when } x = x_1, z = \frac{x_1 - \mu}{\sigma} = z_1 \text{ (say)}$$

$$x = x_2, z = \frac{x_2 - \mu}{\sigma} = z_2 \text{ (say)}$$

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x) dx$$

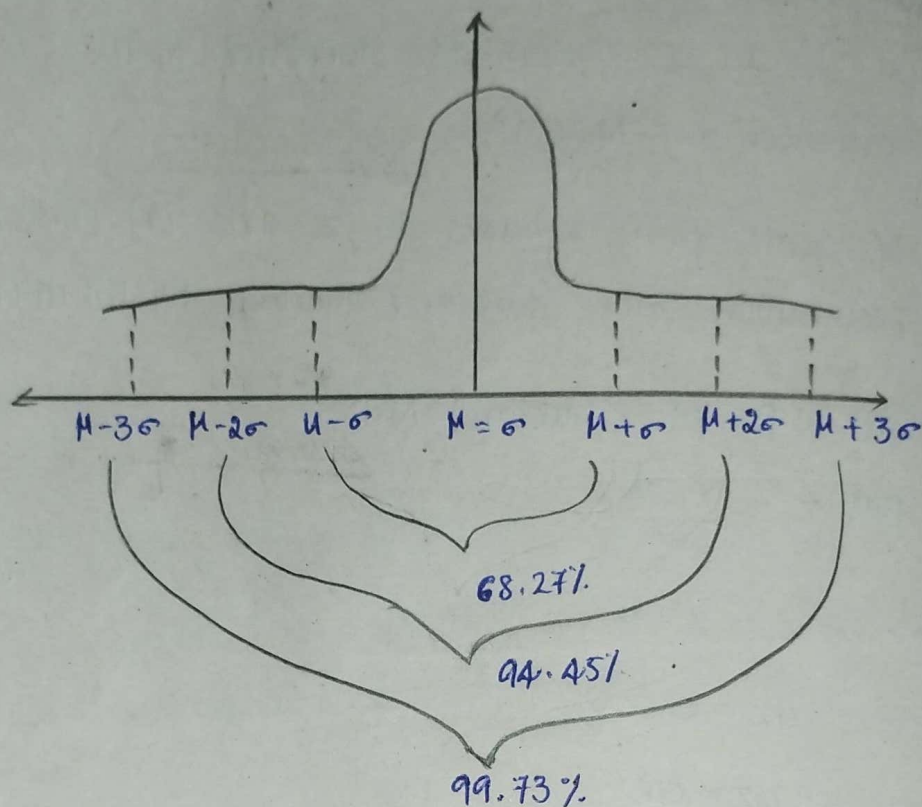
$$= \int_{x_1}^{x_2} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$P(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_2} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{z^2}{2}} (\sigma dz)$$

$$P(z_1 \leq Z \leq z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz$$

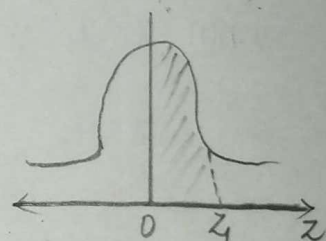
⇒ Above eqn is known as standard N.D,
All these standard N.D table.

> The Area under Normal Curve is distributed as follows.



Note:

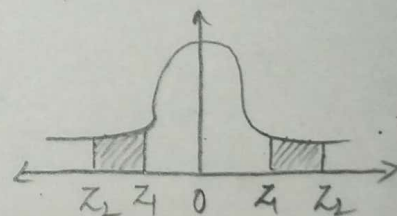
- 1) $P(z_1) = \text{Area from } 0 \text{ to } z_1 = A(0 \text{ to } z_1)$
 $P(z_2) = \text{Area from } 0 \text{ to } z_2 = A(0 \text{ to } z_2)$



- 2) If both z_1 and z_2 are positive (or) negative, then

$$P(z_1 \leq z \leq z_2) = |P(z_2 - z_1)|$$

$$= |A(0 \text{ to } z_2) - A(0 \text{ to } z_1)|$$

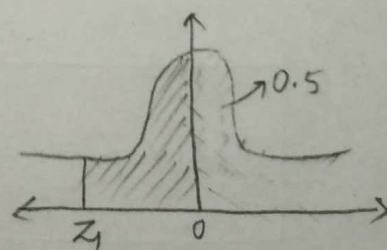


- 3) To find $P(z > z_1)$

- 1) If $z_1 < 0$ then

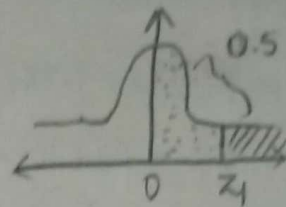
$$P(z > z_1) = 0.5 + A(0 \text{ to } z_1)$$

$$= 0.5 + P(z_1)$$



ii) If $z_1 > 0$, then

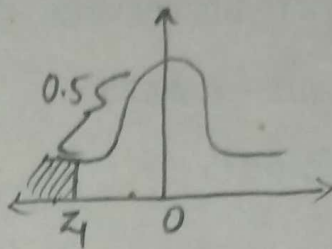
$$P(Z > z_1) = 0.5 - A(0 \text{ to } z_1) \\ = 0.5 - P(z_1)$$



5) To find $P(Z < z_1)$:

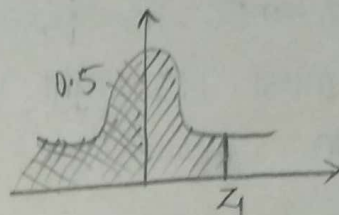
i) If $z_1 < 0$ then

$$P(Z < z_1) = 0.5 - A(0 \text{ to } z_1) \\ = 0.5 - P(z_1)$$



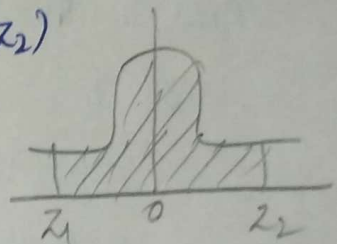
ii) If $z_1 > 0$ then

$$P(Z < z_1) = 0.5 + A(0 \text{ to } z_1) \\ = 0.5 + P(z_1)$$



3) z_1 is -ve, z_2 is +ve

$$P(z_1 < Z < z_2) = A(0 \text{ to } z_1) + A(0 \text{ to } z_2) \\ = P(z_1) + P(z_2)$$



Problems

i) Find the Area under the Normal curve

i) to the left of $z = -1.78$

ii) to the left of $z = 0.56$

iii) to the right of $z = -1.45$

iv) corresponding to $z \geq 2.16$

v) Corresponding to $-0.80 \leq z \leq 1.53$

vi) to the left of $z = -2.52$ and to the right of $z = 1.83$

Qsol.

Note:

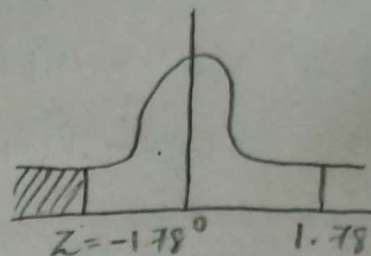
In a Continuous R.V.,

$$P(z_1 < Z < z_2) = P(z_1 \leq Z \leq z_2) = P(z_1 < Z \leq z_2) = \\ P(z_1 \leq Z < z_2)$$

In Normal Distribution all the above are equivalent.

i) To the Left, Area is

$$\begin{aligned} Z = -1.78 &= P(Z < -1.78) \\ &= 0.5 - A(0 \text{ to } -1.78) \\ &= 0.5 - A(0 \text{ to } 1.78) \end{aligned}$$

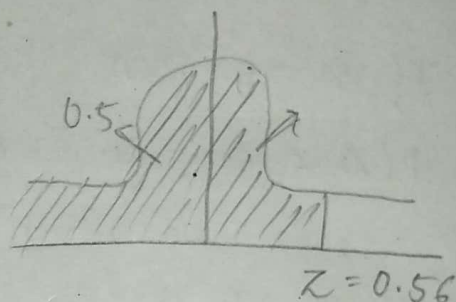


($\because$ Area under the normal curve is Symmetric)

$$\begin{aligned} &= 0.5 - 0.4625 \\ &= 0.0375 \end{aligned}$$

$1.78 = 1.7 + 0.08$

ii) Since $Z = 0.56$ is positive,
So, Z must lie right of
q origin.



Area to the left of q

$$Z = 0.5 + A(0 \text{ to } 0.56)$$

$$= 0.5 + 0.2123 \quad [\because \text{Since from the table}]$$

$$= 0.7123$$

iii) Since $Z = -1.45$ is negative
So, Z must lie left of origin

$\therefore$ Area to the right of q

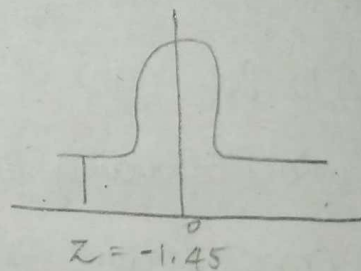
$$Z = 0.5 + A(0 \text{ to } -1.45)$$

$$= 0.5 + A(0 \text{ to } 1.45)$$

$$= 0.5 + 1.4 + 0.05$$

$$= 0.5 + 0.4265$$

$$= 0.9265$$



[$\because$ Since area is symmetric
under normal curve]

iv) corresponding to $z \geq 2.16$

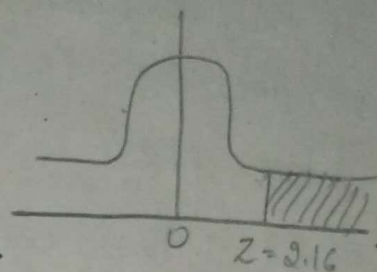
Area corresponding to

$$z \geq 2.16 = 0.5 - A(0 \text{ to } 2.16)$$

$$= 0.5 - 2.1 + 0.06$$

$$= 0.5 - 0.4846$$

$$= 0.0154$$



[$\because$ since from table]

v) Corresponding to

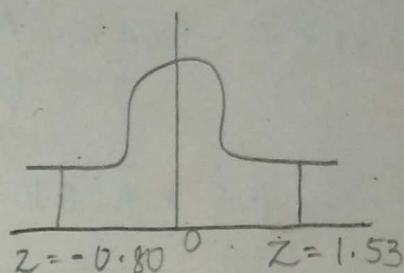
$$z_1 \leq z \leq z_2 \Rightarrow -0.80 \leq z \leq 1.53$$

$$= A(0 \text{ to } -0.80) + A(0 \text{ to } 1.53)$$

$$= A(0 \text{ to } 0.80) + A(0 \text{ to } 1.53)$$

$$= 0.2881 + 0.4370$$

$$= 0.7251$$



[since area is symmetric]

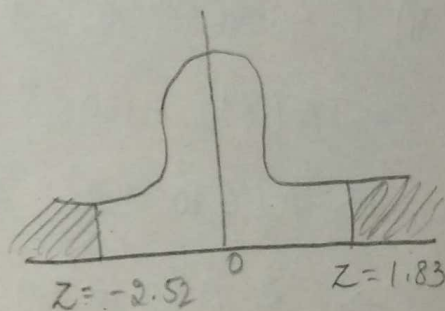
vi) Area to the left of $z = -2.52$
and to the right of $z = 1.83$

$$\Rightarrow 1 - [A(0 \text{ to } -2.52) + A(0 \text{ to } 1.83)]$$

$$\Rightarrow 1 - [A(0 \text{ to } 2.52) + A(0 \text{ to } 1.83)]$$

$$\Rightarrow 1 - [0.4941 + 0.4664]$$

$$\Rightarrow 0.0395$$



[since area is symmetric]

2) If Z is Normally distributed with mean = 0 and variance = 1. Find

i) Prob. $P(Z \geq -1.64)$

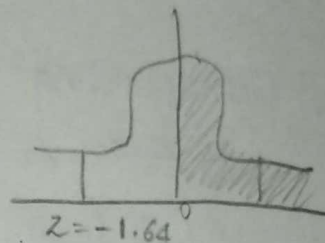
ii) $P(-1.96 \leq Z \leq 1.96)$

iii) $P(Z \leq 1)$

iv) $P(Z \geq 1)$

Sol

i) $P(Z \geq -1.64)$ = Area under normal curve to the right of $Z = -1.64$



$$\Rightarrow 0.5 + A(0 \text{ to } -1.64)$$

$$\Rightarrow 0.5 + A(0 \text{ to } 1.64) \quad [\text{Since area is symmetric}]$$

$$\Rightarrow 0.5 + 0.4495$$

$$\Rightarrow 0.9495$$

ii) $P(-1.96 \leq Z \leq 1.96)$

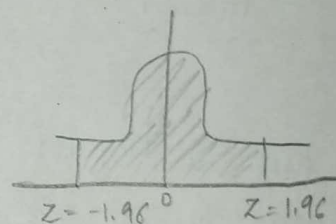
$$\Rightarrow A(0 \text{ to } -1.96) + A(0 \text{ to } 1.96)$$

$$\Rightarrow A(0 \text{ to } 1.96) + A(0 \text{ to } 1.96) \quad [\text{Since area is symmetric}]$$

$$\Rightarrow 2 \cdot A(0 \text{ to } 1.96)$$

$$\Rightarrow 2 \times 0.4750$$

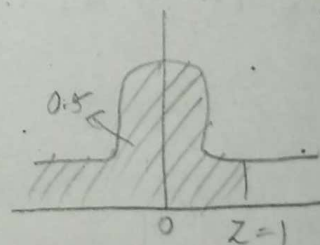
$$\Rightarrow 0.9500$$



iii) $P(Z \leq 1) = 0.5 + A(0 \text{ to } 1)$

$$= 0.5 + 0.3413$$

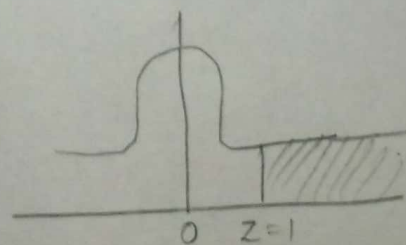
$$= 0.8413$$



iv) $P(Z \geq 1) = 0.5 - A(0 \text{ to } 1)$

$$= 0.5 - 0.3413$$

$$= 0.1587$$



3) Determine the value of z . Such that

i) Area to the right of z is 0.2266

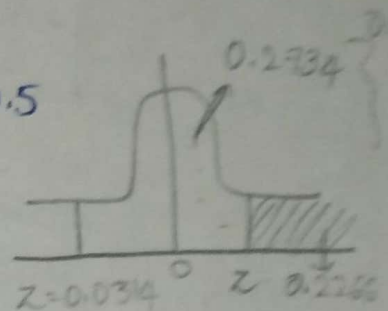
ii) Area to the left of z is 0.0314

iii)

Sol.

i) Since, the area $z = 0.2266$ is less than 0.5

Therefore, $z = 0.2266 < 0.5$, must be positive, and it is on right side.

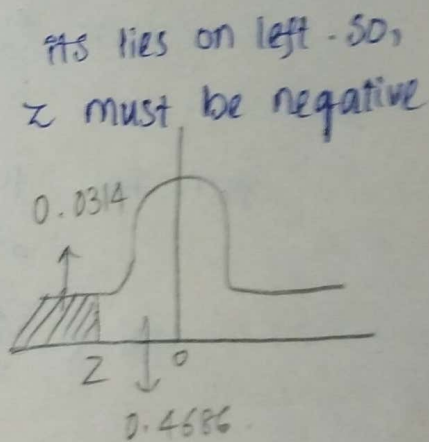


$$\begin{aligned}\text{Area from 0 to } z &= 0.5 - 0.2266 \\ &= 0.2734\end{aligned}$$

Now, for the area 0.2734 from table

$$z = 0.75$$

ii) Area from 0 to $z = 0.5 - 0.0314$, z must be negative
 $= 0.4686$



Now, for the area 0.4686 from table

$$z = -1.86$$

NOTE 1)

i) If z is given directly then the distribution is known as Standard Normal Distribution.

2) If x is given, then the distribution is known as Normal Distribution.

NOTE 2)

Convert the standard Normal distribution into standard normal distribution,

$$Z = \frac{x - \mu}{\sigma}$$

4) If x is normally distributed with mean 70 and standard deviation 16, find,

i) $P(38 \leq x \leq 46)$

ii) $P(82 \leq x \leq 94)$

iii) $P(62 \leq x \leq 86)$

Sol We know that,

To convert Normal distribution to standard Normal distribution,

Given that, $\mu = 70$
 $\sigma = 16$

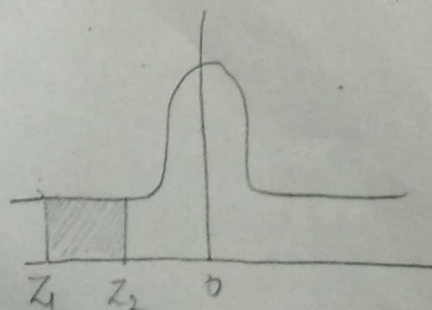
put $z = \frac{x - \mu}{\sigma}$

i) $P(38 \leq x \leq 46)$

when $x = 38$, $z = \frac{38 - 70}{16} \Rightarrow -2 = z_1 \text{ (say)}$

when $x = 46$, $z = \frac{46 - 70}{16} \Rightarrow -1.5 = z_2 \text{ (say)}$

Since both z_1 and z_2 are negative



$$\begin{aligned}
 P(Z_1 \leq Z \leq Z_2) &= |P(Z_1) - P(Z_2)| \\
 &= |A(0 \text{ to } Z_1) - A(0 \text{ to } Z_2)| \\
 &= |A(0 \text{ to } -2) - A(0 \text{ to } -1.5)| \\
 &= |A(0 \text{ to } 2) - A(0 \text{ to } 1.5)| \quad [\text{Since area is symmetric under normal curve}] \\
 &= |0.4772 - 0.4332| \\
 &= 0.044 \quad [\text{Since from Table}]
 \end{aligned}$$

ii) $P(82 \leq X \leq 94)$

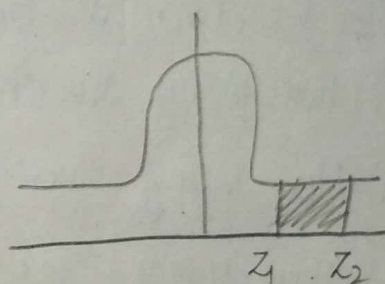
when $x = 82$, $\frac{82 - 70}{16} \Rightarrow 0.75 = Z_1 \text{ (say)}$

when $x = 94$, $\frac{94 - 70}{16} \Rightarrow 1.5 = Z_2 \text{ (say)}$

Since both Z_1 & Z_2 are +ve

$$P(Z_1 \leq Z \leq Z_2) = P(0.75 \leq Z \leq 1.5)$$

$$\begin{aligned}
 &= |P(Z_2) - P(Z_1)| \\
 &= |A(0 \text{ to } Z_2) - A(0 \text{ to } Z_1)| \\
 &= |A(0 \text{ to } 1.5) - A(0 \text{ to } 0.75)| \\
 &= |0.4332 - 0.2734| \\
 &= 0.1598
 \end{aligned}$$



iii) $P(62 \leq X \leq 86)$

when $x = 62$, $\frac{62-70}{16} \Rightarrow -0.5 = z_1$ (say)

when $x = 86$, $\frac{86-70}{16} \Rightarrow 1 = z_2$ (say)

z_1 is -ve & z_2 is +ve.

$$P(z_1 \leq Z \leq z_2) = A(0 \text{ to } z_1) + A(0 \text{ to } z_2)$$

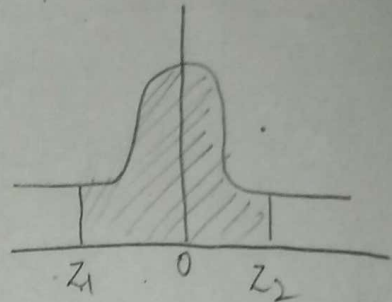
$$= P(z_1) + P(z_2)$$

$$= A(0 \text{ to } -0.5) + A(0 \text{ to } 1)$$

$$= A(0 \text{ to } 0.5) + A(0 \text{ to } 1)$$

$$= 0.1915 + 0.3413$$

$$= 0.5328$$



5) In a sample of 1000 cases, the mean of certain test is 14, standard deviation is 2.5, Assuming the Distribution to be normal, Find

i) How many students score b/w 12 & 15

ii) How many students score above 18

iii) How many students score below 18.

Sol. Given $\mu = 14$
 $\sigma = 2.5$

$$\text{put } z = \frac{x - \mu}{\sigma}$$

i) when $x = 12$, $\frac{12-14}{2.5} = -0.8 = z_1$ (say)

when $x = 15$, $\frac{15-14}{2.5} = 0.4 = z_2$ (say)

z_1 is -ve & z_2 is +ve

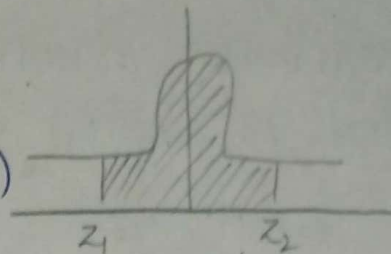
$$P(z_1 \leq Z \leq z_2) = P(z_1) + P(z_2)$$

$$= A(0 \text{ to } -0.8) + A(0 \text{ to } 0.4)$$

$$= A(0 \text{ to } 0.8) + A(0 \text{ to } 0.4)$$

$$= 0.2881 + 0.1554$$

$$= 0.4435$$



Now, No. of students to score b/w 12 & 15 is,
 $\Rightarrow 0.4435 \times 1000$
 $\Rightarrow 443.5 \approx 443$

ii) Above 18

$$\text{when } x=18 \Rightarrow \frac{18-14}{2.5} = 1.6 = z_1 \text{ (say)}$$

Since z_1 is +ve

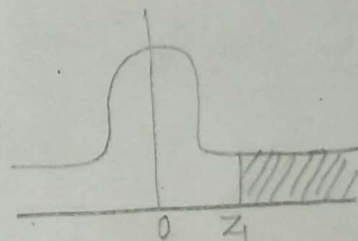
$$P(x > 18) = P(Z > 1.6)$$

$$= 0.5 - P(z_1)$$

$$= 0.5 - A(0 \text{ to } 1.6)$$

$$= 0.5 - 0.4452$$

$$= 0.0548$$



No. of students above 18,

$$0.0548 \times 1000$$

$$\Rightarrow 54.8 \approx 55$$

iii) below 18

$$\text{when } x=18, z_1=1.6 \text{ (say)}$$

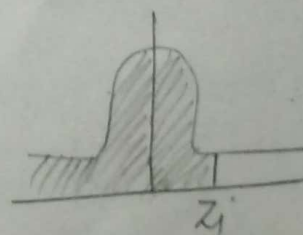
Since z_1 is +ve

$$P(x < 18) = P(Z < 1.6)$$

$$= 0.5 + A(0 \text{ to } 1.6)$$

$$= 0.5 + 0.4452$$

$$= 0.9452$$



No. of students below 18,

$$0.9452 \times 1000$$

$$= 945.2 \approx 945$$

#6) Find the mean and Standard deviation of a Normal distribution in which 7% of the items are under 35 and 89% of the items are under 63.

Sol. Let μ and σ of the N.D.

Given that,

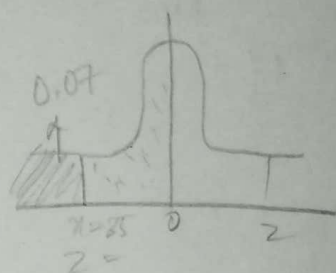
7% of items under 35

$$\text{i.e., } P(X \leq 35) = 7\% \Rightarrow P(X < 35) = 0.07$$

$$\frac{7}{100} = 0.07 < 0.5$$

x is -ve, the corresponding z must be -ve.

$$\text{From fig, Area from 0 to } z = 0.5 - 0.07 = 0.43$$



Now, for the Area 0.43, from table

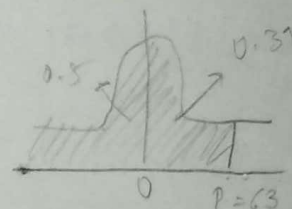
$$z = 1.49$$

Also, given that, $P(X < 63) = 89\%$

$$= 89/100$$

$$= 0.89 > 0.5$$

x is +ve, the corresponding z is also +ve



From fig,

$$\text{Area from 0 to } z = 0.89 - 0.5 = 0.39$$

Now, for the Area 0.39, from table

$$z = 1.23$$

To Convert normal distribution into standard normal distribution, put $z = \frac{x - \mu}{\sigma}$

Q.20 when $x = 35 \Rightarrow Z = 1.48$

$$-1.48 = \frac{35 - \mu}{\sigma}$$

$$\Rightarrow 35 - \mu = -1.48 \sigma$$

$$\Rightarrow \mu - 1.48 \sigma = 35 \quad \text{--- (1)}$$

Q.20 when $x = 63, Z = 1.23$

$$\therefore 1.23 = \frac{63 - \mu}{\sigma}$$

$$\Rightarrow 63 - \mu = 1.23 \sigma$$

$$\Rightarrow \mu + 1.23 \sigma = 63 \quad \text{--- (2)}$$

Solving (1) & (2) we get,

$$\mu = 50.291 \approx 50.3$$

$$\boxed{\mu = 50.3}$$

$$\boxed{\sigma = 10.3}$$

7) Find the mean & standard deviation of a N.D. in which 31% of items are under 45 and 8% are over 64

Sol. Let mean & standard deviation of a Normal distribution are μ and σ

Given that,

$$P(X < 45) = 31\% = \frac{31}{100} = 0.31$$

So, X must be -ve, the corresponding Z is also -ve.

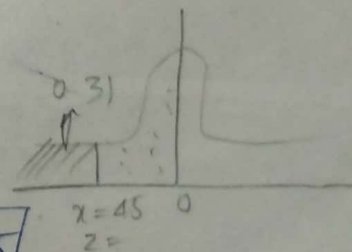
From fig, Area from 0 to Z

$$= 0.5 - 0.3$$

$$= 0.19$$

From table, for the area 0.19 is

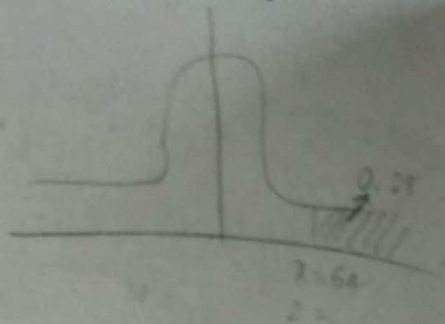
$$\boxed{Z = -0.5}$$



Also, given that

$$P(X > 64) = 8\% = \frac{8}{100} = 0.08 < 0.5$$

So, x must be the +ve, the corresponding z is also +ve.



From fig,

$$\begin{aligned} \text{Area from } 0 \text{ to } z &= 0.5 - 0.08 \\ &= 0.42 \end{aligned}$$

From table, for the area 0.42 then

$$\boxed{z = 1.41}$$

To convert normal distribution to standard normal distribution, put $z = \frac{x - \mu}{\sigma}$

when $x = 45$, $z = -0.5$

$$\therefore -0.5 = \frac{45 - \mu}{\sigma}$$

$$\Rightarrow 45 - \mu = -0.5\sigma$$

$$\Rightarrow \mu - 0.5\sigma = 45 \quad \text{--- (1)}$$

when $x = 64$, $z = 1.41$

$$\therefore 1.41 = \frac{64 - \mu}{\sigma}$$

$$\Rightarrow 64 - \mu = 1.41\sigma$$

$$\Rightarrow \mu + 1.41\sigma = 64 \quad \text{--- (2)}$$

Solving (1) & (2), we get $\mu = 49.97$,

$$\boxed{\mu = 50}$$

$$\sigma = 9.9$$

$$\boxed{\sigma = 10}$$

- 8) Given that the mean heights of students in a class is 158 cms with Standard deviation of 20 cms. Find how many students heights lie b/w 150 cms and 170 cms if there are 100 students in the class.

Sol.

$$\text{Mean } \mu = 158$$

$$\text{S.D } \sigma = 20$$

$$\text{put } z = \frac{x - \mu}{\sigma}$$

$$i) P(150 \leq X \leq 170)$$

$$\text{when } x = 150, \frac{150 - 158}{20} = -0.4 = z_1 \text{ (say)}$$

$$\text{when } x = 170, \frac{170 - 158}{20} = 0.6 = z_2 \text{ (say)}$$

z_1 is -ve & z_2 is +ve

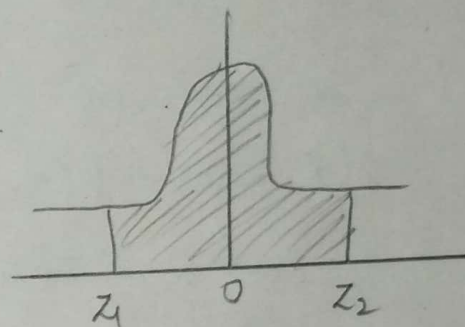
$$P(z_1 \leq Z \leq z_2) = P(z_1) + P(z_2)$$

$$= A(0 \text{ to } -0.4) + A(0 \text{ to } 0.6)$$

$$= A(0 \text{ to } 0.4) + A(0 \text{ to } 0.6)$$

$$= 0.1554 + 0.2257$$

$$= 0.3811$$



Now, No. of students heights lie b/w 150 cms & 170 cms

$$\text{are } 0.3811 \times 100$$

$$38.11$$

Approx. 38 students

10) In a test of 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an avg. lft of 2040 hrs and S.D of 60 hrs. Find

- i) more than 2150 hrs
- ii) less than 1950 hrs
- iii) more than 1920 hrs & less than 2160 hrs.

Sol. Given, total = 2000

$$\mu = 2040$$

$$\sigma = 60$$

$$\text{put } z = \frac{x - \mu}{\sigma}$$

i) $(P > 2150)$

$$\text{when } x = 2150, \frac{2150 - 2040}{60} \Rightarrow 1.83 = z_1 (\text{say})$$

Since z_1 is +ve

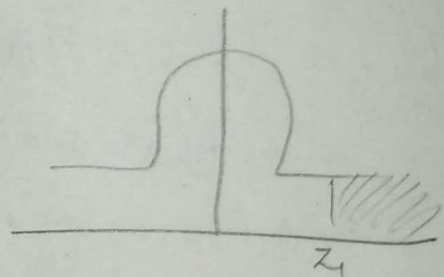
$$P(X > 2150) = P(Z > 1.83)$$

$$= 0.5 - P(z_1)$$

$$= 0.5 - A(0 \text{ to } 1.83)$$

$$= 0.5 - 0.4664$$

$$= 0.0336$$



$$\begin{aligned} \text{more than 2150 hrs is } & 0.0336 \times 2000 \\ & = 67.2 \end{aligned}$$

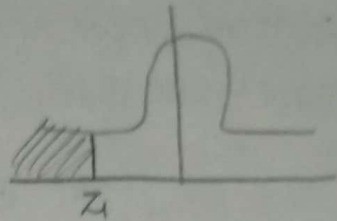
Approx. 67

ii) $P(X < 1950)$

$$\text{when } x = 1950, \frac{1950 - 2040}{60} = -1.5 = z_1 (\text{say})$$

z_1 is -ve

$$\begin{aligned}
 P(X < 1950) &= P(Z < -1.5) \\
 &= 0.5 - A(0 \text{ to } Z_1) \\
 &= 0.5 - P(Z_1) \\
 &= 0.5 - A(0 \text{ to } 1.5) \\
 &= 0.5 - 0.4332 \\
 &= 0.0668
 \end{aligned}$$



less than 1950 hrs is

$\times 2000$

=

$$\Rightarrow 184$$

$$\text{iii) } (1920 \leq Z \leq 2160)$$

$$\text{when } x = 1920, \quad \frac{1920 - 2040}{60} = -2 = Z_1 \text{ (say)}$$

$$\text{when } x = 2160, \quad \frac{2160 - 2040}{60} = 2 \text{ say}$$

Z_1 is -ve & Z_2 is +ve

$$P(Z_1 \leq Z \leq Z_2) = P(Z_1) + P(Z_2)$$

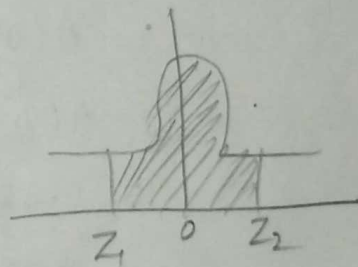
$$= A(0 \text{ to } -2) + A(0 \text{ to } 2)$$

$$= A(0 \text{ to } 2) + A(0 \text{ to } 2)$$

$$= 2A(0 \text{ to } 2)$$

$$= 2 \cdot 0.4772$$

$$= 0.9544$$



more than 1920 hrs & less than 2160 hrs is

$$0.9544 \times 2000$$

$$= 1908.8$$

Approx. 1909

12) The mean height of 500 students is 151 cm. and the S.D is 15 cm. Assuming that the heights are normally distributed Find how many students heights lie b/w 120 cm & 155 cm.

Sol.

$$\text{Mean} = 151 \text{ (M)}$$

$$\text{S.D} = 15 \text{ (}\sigma\text{)}$$

$$\text{total} = 500$$

$$\text{put } z = \frac{x - M}{\sigma}$$

$$i) P(120 \leq Z \leq 155)$$

$$\text{when } x = 120, \frac{120 - 151}{15} = -2.06 = z_1 \text{ (say)}$$

$$\text{when } x = 155, \frac{155 - 151}{15} = 0.26 = z_2 \text{ (say)}$$

z_1 is -ve & z_2 is +ve

$$P(z_1 \leq Z \leq z_2) = P(z_1) + P(z_2)$$

$$= A(0 \text{ to } -2.06) + A(0 \text{ to } 0.26)$$

$$= A(0 \text{ to } 2.06) + A(0 \text{ to } 0.26)$$

$$= 0.4803 + 0.1026$$

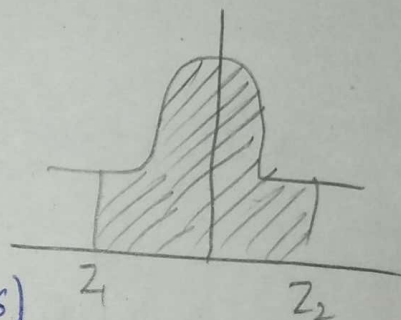
$$= 0.5829$$

Students heights lies b/w 120 cm & 155 cm is

$$0.5829 \times 500$$

$$291.45$$

Approx. 291



14) The mean & S.D of marks obtained by 1000 students in exam are respectively 34.5 & 16.5. Assuming the normality of distribution. Find marks blw 30 & 60.

Sol. Total = 1000

mean = 34.5

S.D = 16.5

$$\text{put } z = \frac{x - M}{\sigma}$$

$$P(30 \leq Z \leq 60)$$

$$\text{when } x = 30, \quad \frac{30 - 34.5}{16.5} = -0.27 = z_1 \text{ (say)}$$

$$\text{when } x = 60, \quad \frac{60 - 34.5}{16.5} = 1.54 = z_2$$

z_1 is -ve & z_2 is +ve

$$P(z_1 \leq Z \leq z_2) = P(z_1) + P(z_2)$$

$$A(0 \text{ to } -0.27) + A(0 \text{ to } 1.54)$$

$$A(0 \text{ to } 0.27) + A(0 \text{ to } 1.54)$$

$$0.1064 + 0.4394$$

$$= 0.5458$$

$$\Rightarrow \text{blw } 30 \text{ \& } 60 \text{ are } 0.5458 \times 1000 \Rightarrow 545.8$$

$$\Rightarrow 546 \text{ students}$$

